

Magnetic Dirichlet Laplacian with radially symmetric magnetic field

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Abstract. The aim of the paper is to derive spectral estimates on the eigenvalue moments of the magnetic Dirichlet Laplacian defined on the two-dimensional disk with a radially symmetric magnetic field.

1. INTRODUCTION

Let us consider a particle in a domain Ω in \mathbb{R}^2 in the presence of a magnetic field B . We define the 2-dimensional magnetic Laplacian associated to this particle as follows:

Let A be a magnetic potential associated to B , i.e. a smooth real valued-function on $\Omega \subset \mathbb{R}^2$ verifying $\text{rot}(A) = B$. The magnetic Dirichlet Laplacian is initially defined on $C_0^\infty(\Omega)$ by $H_\Omega(A) = (i\nabla + A)^2$.

Under the assumptions that Ω is a bounded domain and that A satisfies mild regularity conditions, which means that in particular the magnetic field $B \in L_{\text{loc}}^\infty(\mathbb{R}^2)$ and the corresponding magnetic potential $A \in L^\infty(\Omega)$, the magnetic Sobolev norm $\|(i\nabla + A)u\|_{L^2(\Omega)}$, $u \in \mathcal{H}_0^1(\Omega)$, is closed and equivalent to the non-magnetic one, so the self-adjoint Friedrich's extension has a purely discrete spectrum as in the non-magnetic case.

In the paper we also consider the case when the magnetic field grows to infinity as the variable approaches the boundary and has a non zero infimum

$$B(z) \rightarrow \infty \quad \text{as } z \rightarrow \partial\Omega \quad \text{and} \quad K := \inf B(z) > 0. \quad (1.1)$$

In view of the lower bound

$$(H_\Omega(A)(u), u)_{L^2(\Omega)} \geq \int_\Omega B(z)|u|^2(z) \, dz,$$

one again can construct the Friedrich's extension of $H_\Omega(A)$ initially defined on $C_0^\infty(\Omega)$. Moreover, it still has a purely discrete spectrum— [T12].

For simplicity, we will use for the Friedrich's extension the same symbol $H_\Omega(A)$, and we shall denote the increasingly ordered sequence of its eigenvalues by $\lambda_k = \lambda_k(\Omega, A)$.

The purpose of this paper is to establish bounds of the eigenvalue moments of such operators. Let us recall the following bound which was proved by Berezin, Li and Yau for non-magnetic Dirichlet Laplacians on a domain Ω in \mathbb{R}^d – [Be72a, Be72b, LY83],

$$\sum_k (\Lambda - \lambda_k(\Omega, 0))_+^\sigma \leq L_{\sigma,d}^{\text{cl}} |\Omega| \Lambda^{\sigma + \frac{d}{2}} \quad \text{for any } \sigma \geq 1 \text{ and } \Lambda > 0, \quad (1.2)$$

where $|\Omega|$ is the volume of Ω , and the constant on the right-hand side,

$$L_{\sigma,d}^{\text{cl}} = \frac{\Gamma(\sigma + 1)}{(4\pi)^{\frac{d}{2}} \Gamma(\sigma + 1 + d/2)}, \quad (1.3)$$

is optimal. Moreover, for $0 \leq \sigma < 1$, the bound (1.2) still exists, but with another constant on the right-hand side – [La97]

$$\sum_k (\Lambda - \lambda_k(\Omega, 0))_+^\sigma \leq 2 \left(\frac{\sigma}{\sigma + 1} \right)^\sigma L_{\sigma,d}^{\text{cl}} |\Omega| \Lambda^{\sigma + \frac{d}{2}}, \quad 0 \leq \sigma < 1. \quad (1.4)$$

In the magnetic case, in view of the pointwise diamagnetic inequality which means that under the rather general assumptions on the magnetic potentials [LL01]

$$|\nabla|u(x)|| \leq |(i\nabla + A)u(x)| \quad \text{for a.a. } x \in \Omega,$$

we get that $\lambda_1(\Omega, A) \geq \lambda_1(\Omega, 0)$. However, the estimate $\lambda_j(\Omega, A) \geq \lambda_j(\Omega, 0)$ fails in general if $j \geq 2$. Let us mention that, nevertheless, momentum estimates are still valid for some values of the parameters. In particular, it was shown [LW00] that the sharp bound (1.2) holds true for arbitrary magnetic fields provided $\sigma \geq \frac{3}{2}$, and for constant magnetic fields if $\sigma \geq 1$ – [ELV00]. In the two-dimensional case the bound (1.4) holds true for constant magnetic fields if $0 \leq \sigma < 1$, and the constant on the right-hand side cannot be improved – [FLW09].

In the present work we study the magnetic Dirichlet Laplacian $H_\Omega(A)$ defined on the two-dimensional disk Ω centered in zero and with radius $r_0 > 0$, with a radially symmetric magnetic field $B(x) = B(|x|) \geq 0$. Our aim is to extend a sufficiently precise Berezin type inequality to this situation. A similar problem was studied recently in [BEKW16], but under very strong restrictions on the growth of the magnetic field.

Let us also mention that some estimates on the counting function of the eigenvalues of the magnetic Dirichlet Laplacian on a disk were established in [T12], in the case where the field is radial and satisfies some growth condition near the boundary.

2. MAIN RESULTS

Before stating the results we define the following one-dimensional operators $l(B)$ and $\widetilde{l}(B)$ in $L^2((0, r_0), 2\pi r dr)$ associated with the closures of the quadratic forms

$$Q(l(B))[u] = \int_0^{r_0} r \left| \frac{du}{dr} \right|^2 dr + \int_0^{r_0} \frac{1}{r} \left(\int_r^1 s B(s) ds \right)^2 |u|^2 dr,$$

$$Q(\widetilde{l}(B))[u] = \int_0^{r_0} r \left| \frac{du}{dr} \right|^2 dr + \int_0^{r_0} \frac{1}{r} \left(\int_0^r s B(s) ds \right)^2 |u|^2 dr$$

defined originally on $C_0^\infty(0, r_0)$, and acting on their domain as

$$\begin{aligned} l(B) &= -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{1}{r^2} \left(\int_r^{r_0} sB(s) ds \right)^2, \\ \widetilde{l}(B) &= -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{1}{r^2} \left(\int_0^r sB(s) ds \right)^2. \end{aligned} \quad (2.1)$$

The following theorem holds true:

Theorem 2.1. *Let $H_\Omega(A)$ be the magnetic Dirichlet Laplacian on the disk Ω of radius equal to r_0 centered at the origin with a radial magnetic field $B(x) = B(|x|) \geq 0$. Let us assume the validity of the mild regularity conditions for the magnetic potential A discussed in Introduction or the validity of (1.1). Moreover, let $\int_0^{r_0} sB(s) ds < \infty$.*

If $\int_0^{r_0} sB(s) ds \notin \mathbb{Z}$, then for any $\Lambda \geq 0$ and $\sigma \geq 3/2$, the following inequality holds

$$\begin{aligned} \text{tr}(\Lambda - H_\Omega(A))_+^\sigma &\leq \frac{1}{2} \text{tr} \left(\Lambda - \left(-\Delta_D^\Omega + \frac{1}{x^2 + y^2} \left(\int_{\sqrt{x^2+y^2}}^{r_0} sB(s) ds \right)^2 \right) \right)_+^\sigma \\ &\quad + \frac{1}{2} \text{tr} \left(\Lambda - \left(-\Delta_D^\Omega + \frac{1}{x^2 + y^2} \left(\int_0^{\sqrt{x^2+y^2}} sB(s) ds \right)^2 \right) \right)_+^\sigma \\ &+ \frac{2L_{\sigma,1}^{\text{cl}} r_0^{2\sigma+1}}{2\sigma + 1} \left[\int_0^{r_0} sB(s) ds \right] \Lambda^{\sigma+1/2} + \frac{1}{2} \text{tr}(\Lambda - l(B))_+^\sigma + \frac{1}{2} \text{tr}(\Lambda - \widetilde{l}(B))_+^\sigma \end{aligned}$$

If $\int_0^{r_0} sB(s) ds \in \mathbb{Z}$, then for any $\Lambda \geq 0$ and $\sigma \geq 3/2$, the following inequality holds

$$\begin{aligned} \text{tr}(\Lambda - H_\Omega(A))_+^\sigma &\leq \frac{1}{2} \text{tr} \left(\Lambda - \left(-\Delta_D^\Omega + \frac{1}{x^2 + y^2} \left(\int_{\sqrt{x^2+y^2}}^{r_0} sB(s) ds \right)^2 \right) \right)_+^\sigma \\ &\quad + \frac{1}{2} \text{tr} \left(\Lambda - \left(-\Delta_D^\Omega + \frac{1}{x^2 + y^2} \left(\int_0^{\sqrt{x^2+y^2}} sB(s) ds \right)^2 \right) \right)_+^\sigma \\ &+ \frac{2L_{\sigma,1}^{\text{cl}} r_0^{2\sigma+1}}{2\sigma + 1} \left[\int_0^{r_0} sB(s) ds \right] \Lambda^{\sigma+1/2} - \frac{1}{2} \text{tr}(\Lambda - l(B))_+^\sigma + \frac{1}{2} \text{tr}(\Lambda - \widetilde{l}(B))_+^\sigma. \end{aligned}$$

where the operators $l(B)$ and $\widetilde{l}(B)$ are defined in (2.1) and $L_{\sigma,1}^{\text{cl}}$ is the semiclassical constant given by (1.3).

Proof. We use the standard partial wave decomposition – [E96]

$$L^2(\Omega) = \bigoplus_{m=-\infty}^{\infty} L^2((0, r_0), 2\pi r dr)$$

and

$$H_\Omega(A) = \bigoplus_{m=-\infty}^{\infty} h_m(B),$$

where the operators $h_m(B)$ in $L^2((0, r_0), 2\pi r dr)$ are associated with the closures of the quadratic forms

$$Q(h_m(B))[u] = \int_0^{r_0} r \left| \frac{du}{dr} \right|^2 dr + \int_0^{r_0} \left(\frac{m}{r} - \frac{1}{r} \int_0^r sB(s) ds \right)^2 r |u|^2 dr,$$

defined originally on $C_0^\infty(0, r_0)$, and acting on their domain as

$$h_m(B) = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \left(\frac{m}{r} - \frac{1}{r} \int_0^r sB(s) ds \right)^2.$$

Let us first consider the case where $m \geq \left[\int_0^{r_0} sB(s) ds \right] + 1$. Then

$$\begin{aligned} \bigoplus_{m \geq \left[\int_0^{r_0} sB(s) ds \right] + 1} h_m(B) &= \bigoplus_{k \geq 1} \left(-\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{1}{r^2} \left(\left[\int_0^{r_0} sB(s) ds \right] + k - \int_0^r sB(s) ds \right)^2 \right) \\ &= \bigoplus_{k \geq 1} \left(-\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{1}{r^2} \left(\int_0^{r_0} sB(s) ds - \left\{ \int_0^{r_0} sB(s) ds \right\} + k - \int_0^r sB(s) ds \right)^2 \right) \\ &= \bigoplus_{k \geq 1} \left(-\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{1}{r^2} \left(\int_r^{r_0} sB(s) ds - \left\{ \int_0^{r_0} sB(s) ds \right\} + k \right)^2 \right) \quad (2.2) \end{aligned}$$

If $\int_0^{r_0} sB(s) ds \notin \mathbb{Z}$ then we get from (2.2)

$$\begin{aligned} \bigoplus_{m \geq \left[\int_0^{r_0} sB(s) ds \right] + 1} h_m(B) &\geq \bigoplus_{n \geq 0} \left(-\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{1}{r^2} \left(\int_r^{r_0} sB(s) ds + n \right)^2 \right) \\ &\geq \bigoplus_{n \geq 0} \left(-\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{n^2}{r^2} + \frac{1}{r^2} \left(\int_r^{r_0} sB(s) ds \right)^2 \right). \end{aligned}$$

Therefore for any $\Lambda \geq 0$ and $\sigma \geq 0$

$$\operatorname{tr} \left(\Lambda - \bigoplus_{m \geq \left[\int_0^{r_0} sB(s) ds \right] + 1} h_m(B) \right)_+^\sigma \leq \operatorname{tr} \left(\Lambda - \bigoplus_{n=0}^\infty l_n(B) \right)_+^\sigma, \quad (2.3)$$

where the operators $l_n(B)$ in $L^2((0, r_0), 2\pi r dr)$ are associated with the closures of the quadratic forms

$$Q(l_n(B))[u] = \int_0^{r_0} r \left| \frac{du}{dr} \right|^2 dr + \int_0^1 \left(\frac{n^2}{r} + \frac{1}{r} \left(\int_r^{r_0} sB(s) ds \right)^2 \right) |u|^2 dr,$$

defined originally on $C_0^\infty(0, r_0)$, and acting on their domain as

$$l_n(B) = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{n^2}{r^2} + \frac{1}{r^2} \left(\int_r^{r_0} sB(s) ds \right)^2.$$

On the other hand, if $\int_0^{r_0} sB(s) ds \in \mathbb{Z}$ then (2.2) writes

$$\begin{aligned} \bigoplus_{m \geq [\int_0^{r_0} sB(s) ds] + 1} h_m(B) &= \bigoplus_{k \geq 1} \left(-\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{1}{r^2} \left(\int_r^{r_0} sB(s) ds + k \right)^2 \right) \\ &\geq \bigoplus_{k \geq 1} \left(-\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{k^2}{r^2} + \frac{1}{r^2} \left(\int_r^{r_0} sB(s) ds \right)^2 \right) \end{aligned}$$

and, similarly,

$$\mathrm{tr} \left(\Lambda - \bigoplus_{m \geq [\int_0^{r_0} sB(s) ds] + 1} h_m(B) \right)_+^\sigma \leq \mathrm{tr} \left(\Lambda - \bigoplus_{k=1}^\infty l_k(B) \right)_+^\sigma. \quad (2.4)$$

Using the following trace symmetry of the operators $l_n(B)$ with respect to n

$$\mathrm{tr} \left(\Lambda - \bigoplus_{n>0} l_n(B) \right)_+^\sigma = \mathrm{tr} \left(\Lambda - \bigoplus_{n<0} l_n(B) \right)_+^\sigma$$

we write

$$\begin{aligned} \mathrm{tr} \left(\Lambda - \bigoplus_{n \in \mathbb{Z}} l_n(B) \right)_+^\sigma &= \mathrm{tr} \left(\Lambda - \bigoplus_{n>0} l_n(B) \right)_+^\sigma + \mathrm{tr} \left(\Lambda - \bigoplus_{n<0} l_n(B) \right)_+^\sigma + \mathrm{tr} (\Lambda - l(B))_+^\sigma \\ &= 2 \mathrm{tr} \left(\Lambda - \bigoplus_{n>0} l_n(B) \right)_+^\sigma + \mathrm{tr} (\Lambda - l(B))_+^\sigma, \end{aligned}$$

where the operator $l(B)$ is defined in (2.1), so that inequality (2.3) implies

$$\mathrm{tr} \left(\Lambda - \bigoplus_{m \geq [\int_0^{r_0} sB(s) ds] + 1} h_m(B) \right)_+^\sigma \leq \frac{1}{2} \mathrm{tr} \left(\Lambda - \bigoplus_{n \in \mathbb{Z}} l_n(B) \right)_+^\sigma + \frac{1}{2} \mathrm{tr} (\Lambda - l(B))_+^\sigma, \quad (2.5)$$

if $\int_0^{r_0} sB(s) ds \notin \mathbb{Z}$,

and inequality (2.4) implies

$$\mathrm{tr} \left(\Lambda - \bigoplus_{m \geq [\int_0^{r_0} sB(s) ds] + 1} h_m(B) \right)_+^\sigma \leq \frac{1}{2} \mathrm{tr} \left(\Lambda - \bigoplus_{n \in \mathbb{Z}} l_n(B) \right)_+^\sigma - \frac{1}{2} \mathrm{tr} (\Lambda - l(B))_+^\sigma \quad (2.6)$$

if $\int_0^{r_0} sB(s) ds \in \mathbb{Z}$.

Applying the partial wave decomposition for the two-dimensional Schrödinger operator $-\Delta_D^\Omega + \frac{1}{x^2+y^2} \left(\int_{\sqrt{x^2+y^2}}^{r_0} sB(s) ds \right)^2$ we obtain from the inequalities (2.5) and (2.6) that, for any

$\sigma \geq 0$

$$\begin{aligned} \operatorname{tr} \left(\Lambda - \bigoplus_{m \geq [\int_0^{r_0} sB(s) ds] + 1} h_m(B) \right)_+^\sigma &\leq \frac{1}{2} \operatorname{tr} \left(\Lambda - \left(-\Delta_D^\omega + \frac{1}{x^2 + y^2} \left(\int_0^{r_0} \frac{sB(s) ds}{\sqrt{x^2 + y^2}} \right)^2 \right) \right)_+^\sigma \\ &\quad + \frac{1}{2} \operatorname{tr} (\Lambda - l(B))_+^\sigma, \end{aligned} \quad (2.7)$$

if $\int_0^{r_0} sB(s) ds \notin \mathbb{Z}$, and that

$$\begin{aligned} \operatorname{tr} \left(\Lambda - \bigoplus_{m \geq [\int_0^{r_0} sB(s) ds] + 1} h_m(B) \right)_+^\sigma &\leq \frac{1}{2} \operatorname{tr} \left(\Lambda - \left(-\Delta_D^\omega + \frac{1}{x^2 + y^2} \left(\int_0^{r_0} \frac{sB(s) ds}{\sqrt{x^2 + y^2}} \right)^2 \right) \right)_+^\sigma \\ &\quad - \frac{1}{2} \operatorname{tr} (\Lambda - l(B))_+^\sigma, \end{aligned} \quad (2.8)$$

if $\int_0^{r_0} sB(s) ds \in \mathbb{Z}$.

Now we move on to the case where $m \leq 0$. Since

$$h_m(B) \geq -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{m^2}{r^2} + \frac{1}{r^2} \left(\int_0^r sB(s) ds \right)^2$$

then, repeating the same ideas as before we arrive to

$$\begin{aligned} \operatorname{tr} \left(\Lambda - \bigoplus_{m \leq 0} h_m(B) \right)_+^\sigma &= \operatorname{tr} \left(\Lambda - \bigoplus_{m < 0} h_m(B) \right)_+^\sigma + \operatorname{tr} (\Lambda - \widetilde{l}(B))_+^\sigma \\ &\leq \frac{1}{2} \operatorname{tr} \left(\Lambda - \left(-\Delta_D^\omega + \frac{1}{x^2 + y^2} \left(\int_0^{\sqrt{x^2 + y^2}} sB(s) ds \right)^2 \right) \right)_+^\sigma + \frac{1}{2} \operatorname{tr} (\Lambda - \widetilde{l}(B))_+^\sigma, \end{aligned} \quad (2.9)$$

where $\widetilde{l}(B)$ is given by (2.1).

Finally let $1 \leq m \leq [\int_0^{r_0} sB(s) ds]$. We follow the method of [T12]. Performing the change of the variables $r = r_0 e^t$ in the quadratic form $Q_m(h_m(B) - \Lambda)$ corresponding to $h_m(B) - \Lambda$

$$Q_m(h_m(B) - \Lambda) = \int_0^{r_0} r |u'|^2 dr + \int_0^{r_0} r \left(\left(\frac{m}{r} - \frac{1}{r} \int_0^r sB(s) ds \right)^2 - \Lambda \right) |u|^2 dr \quad (2.10)$$

and defining w by $w(t) = u(r_0 e^t)$ we transfer (2.10) to

$$\widetilde{Q}_m(B) = \int_{-\infty}^0 |w'|^2 dt + \int_{-\infty}^0 \left(\left(m - \int_0^{r_0 e^t} sB(s) ds \right)^2 - \Lambda r_0^2 e^{2t} \right) |w|^2 dt, \quad (2.11)$$

where $\widetilde{Q}_m(B)$ is defined on $\mathcal{H}_0^1(-\infty, 0)$ and corresponds to the one-dimensional operator

$$g_{m,B} = -\frac{d^2}{dt^2} + \left(m - \int_0^{r_0 e^t} sB(s) ds \right)^2 - \Lambda r_0^2 e^{2t}.$$

It is easy to notice that for any positive $\varepsilon < 1$

$$g_{m,B} \geq g_B + (1 - \varepsilon)m^2, \quad (2.12)$$

where the operator

$$g_B = -\frac{d^2}{dt^2} - \left(\frac{1}{\varepsilon} - 1\right) \left(\int_0^{r_0 e^t} sB(s) ds \right)^2 - \Lambda r_0^2 e^{2t}$$

is defined on $\mathcal{H}_0^1(-\infty, 0)$.

Let $\{\mu_k(B)\}_{k=1}^\infty$ be the set of the negative eigenvalues of $g_{m,B}$. Then due to the minimax principle (2.12) implies

$$\begin{aligned} \operatorname{tr} \left(\Lambda - \bigoplus_{m=1}^{\lfloor \int_0^{r_0} sB(s) ds \rfloor} h_m(B) \right)_+^\sigma &= \sum_{m=1}^{\lfloor \int_0^{r_0} sB(s) ds \rfloor} \operatorname{tr} (\Lambda - h_m(B))_+^\sigma \\ &= \sum_{m=1}^{\lfloor \int_0^{r_0} sB(s) ds \rfloor} \operatorname{tr} (g_{m,B})_-^\sigma \leq \sum_{m=1}^{\lfloor \int_0^{r_0} sB(s) ds \rfloor} \operatorname{tr} (g_B + (1 - \varepsilon)m^2)_-^\sigma \\ &\leq \sum_{m=1}^{\lfloor \int_0^{r_0} sB(s) ds \rfloor} \sum_{\mu_k(B) + (1 - \varepsilon)m^2 \leq 0} |\mu_k(B) + (1 - \varepsilon)m^2|^\sigma \\ &\leq \sum_{|\mu_k(B)| \leq (1 - \varepsilon) \lfloor \int_0^{r_0} sB(s) ds \rfloor^2} \sum_{1 \leq |m| \leq \frac{1}{\sqrt{1 - \varepsilon}} \sqrt{|\mu_k(B)|}} |\mu_k(B) + (1 - \varepsilon)m^2|^\sigma \\ &+ \sum_{|\mu_k(B)| > (1 - \varepsilon) \lfloor \int_0^{r_0} sB(s) ds \rfloor^2} \sum_{1 \leq |m| \leq \lfloor \int_0^{r_0} sB(s) ds \rfloor} |\mu_k(B) + (1 - \varepsilon)m^2|^\sigma \\ &\leq 2 \left[\int_0^{r_0} sB(s) ds \right] \sum_k |\mu_k(B)|^\sigma \end{aligned} \quad (2.13)$$

Let us extend the potential $-\left(\frac{1}{\varepsilon} - 1\right) \left(\int_0^{r_0 e^t} sB(s) ds \right)^2 - \Lambda r_0^2 e^{2t}$ to \mathbb{R} by zero and denote the corresponding one dimensional Schrödinger operator by \widetilde{g}_B . (We omit the dependence on ε in the notation for simplicity). Since $C_0^\infty(-\infty, 0) \subset C_0^\infty(\mathbb{R})$ then

$$\sum_k |\mu_k(B)|^\sigma \leq \sum_k |\nu_k(B)|^\sigma, \quad (2.14)$$

where $\{\nu_k(B)\}_{k=1}^\infty$ are the negative eigenvalues of \widetilde{g}_B .

Applying the trace formulae – [LT76] for any $\sigma \geq 3/2$ we get

$$\begin{aligned} \sum_k |\nu_k(B)|^\sigma &\leq L_{\sigma,1}^{\text{cl}} \int_{-\infty}^0 \left(\left(\frac{1}{\varepsilon} - 1 \right) \left(\int_0^{r_0 e^t} sB(s) ds \right)^2 + \Lambda r_0^2 e^{2t} \right)^{\sigma+1/2} dt \\ &= L_{\sigma,1}^{\text{cl}} \int_0^{r_0} \left(\left(\frac{1}{\varepsilon} - 1 \right) \left(\int_0^r sB(s) ds \right)^2 + \Lambda r^2 \right)^{\sigma+1/2} \frac{1}{r} dr. \end{aligned}$$

After passing to the limit $\varepsilon \rightarrow 1$ the above inequality yields

$$\sum_k |v_k(B)|^\sigma \leq \frac{r_0^{2\sigma+1}}{2\sigma+1} L_{\sigma,1}^{\text{cl}} \Lambda^{\sigma+1/2}. \quad (2.15)$$

By virtue of the estimates (2.13)-(2.15) we obtain

$$\text{tr} \left(\Lambda - \bigoplus_{m=1}^{\lfloor \int_0^{r_0} sB(s) ds \rfloor} h_m(B) \right)_+^\sigma \leq \frac{2L_{\sigma,1}^{\text{cl}} r_0^{2\sigma+1}}{2\sigma+1} \left[\int_0^{r_0} sB(s) ds \right] \Lambda^{\sigma+1/2},$$

which together with (2.7)- (2.9) establishes the theorems. \square

Remark 2.2. Let us assume that $\int_0^{r_0} sB(s) ds < 1$. In view of Theorem 2.1 the threshold of the spectrum of the corresponding magnetic Laplacian can be estimated from below by the minimum of the spectral thresholds of one-dimensional operators $l(B)$ and $\tilde{l}(B)$ and the two-dimensional Schrödinger operators with the potentials $\frac{1}{x^2+y^2} \left(\int_0^{r_0} \frac{sB(s) ds}{\sqrt{x^2+y^2}} \right)^2$ and $\frac{1}{x^2+y^2} \left(\int_0^{\sqrt{x^2+y^2}} sB(s) ds \right)^2$, which is not possible to obtain from a standard estimate (1.2).

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